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Generalized Ornstein–Uhlenbeck processes and associated self-similar processes

S C Lim and S V Muniandy

School of Applied Physics, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia

E-mail: sclim@pkrisc.cc.ukm.my

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Abstract
We consider three types of generalized Ornstein–Uhlenbeck processes: the stationary process obtained from the Lamperti transformation of fractional Brownian motion, the process with stretched exponential covariance and the process obtained from the solution of the fractional Langevin equation. These stationary Gaussian processes have many common properties, such as the fact that their local covariances share a similar structure and they exhibit identical spectral densities at large frequency limit. In addition, the generalized Ornstein–Uhlenbeck processes can be shown to be local stationary representations of fractional Brownian motion. Two new self-similar Gaussian processes, in addition to fractional Brownian motion, are obtained by applying the (inverse) Lamperti transformation to the generalized Ornstein–Uhlenbeck processes. We study some of the properties of these self-similar processes such as the long-range dependence. We give a simulation of their sample paths based on numerical Karhunan–Loeve expansion.

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1. Introduction

Brownian motion (BM) $X(t)$ and Ornstein–Uhlenbeck (OU) process $Y(t)$ are two of the most well studied and widely applied stochastic processes. Despite certain differences in their properties (notably the OU process is stationary whereas BM is non-stationary and self-similar), these two processes have the same local behaviour. In particular, both processes are nowhere differentiable and they are Holder continuous of order $1/2$. The OU process can be regarded as the stationary analogue of BM. Its spectral density $S(\omega) = (\omega^2 + a^2)^{-1}$, $a > 0$, becomes the spectral density of BM in the limit $a \to 0$ or the high-frequency limit. The OU process can be seen as a time-changed BM; the two process are connected by a time transformation $Y(t) = \mathrm{e}^{-at} X(\mathrm{e}^{2at})/\sqrt{2a}$ or conversely $X(t) = \sqrt{t}\, Y\left(\frac{1}{2a} \log t\right)$. This time transformation is a special case of the Lamperti transformation [1], which provides a one-to-one correspondence between stationary and self-similar processes. In fact, the similarity in the local properties of these two processes can be attributed to this time transformation relation.

BM has been successfully extended to fractional Brownian motion (FBM) [2], which has found wide applications in hydrology, condensed matter physics, biological physics, econophysics, telecommunication networks, geoscience and other fields [3–7]. A natural question arises whether there exists a generalized OU process which is related to FBM in a similar way as the OU process is linked to BM. If a generalized OU process is defined as a Gaussian stationary process characterized by a parameter, say α, so that it becomes an OU process when $\alpha = 1$, then there exist several such processes. In this paper, we consider three types of generalized OU processes based on the solution of the fractional Langevin equation, the Lamperti transformation of FBM and the process with stretched exponential covariance. Although it seems only one of these processes can be regarded as the genuine stationary analogue of FBM, all three processes have many properties in common. They have the same spectral density as that of FBM in the asymptotic high-frequency limit. It can also be shown that these generalized OU processes can be regarded as the local stationary representations of FBM [8].

So far FBM is the most widely used Gaussian self-similar process for modelling phenomena with scaling and long memory. Although it is one of the simplest self-similar processes with good properties such as stationary increments, FBM has its limitations, in particular its properties are determined by a single parameter, the Hurst index. In view of the increasing interest in the modelling of long memory phenomena recently [9], it would be useful to consider more general self-similar processes which are characterized by more than one parameter but still preserve some of the good properties (or their weaker versions) of FBM. For this purpose, we apply the Lamperti transformation to the generalized OU processes to obtain two new self-similar processes. We study the properties of these processes such as long-range dependence (LRD) and weaker stationarity of their increments. Computer simulations of the spectral densities of generalized OU processes, their sample paths as well as those of the self-similar counterparts are also given.

2. Generalized OU processes

In this section, we consider three types of generalized OU processes: (a) the stationary process obtained by applying the Lamperti transformation to FBM; (b) the process with stretched exponential covariance; (c) the process obtained from the solution of the fractional Langevin equation.

2.1. Lamperti transformation of FBM

FBM is a Gaussian process defined by [2]

$$X_1(t) = \frac{1}{\Gamma(H+1/2)}\left[\int_{-\infty}^{0} (t-s)^{H-1/2} - (-s)^{H-1/2}\,\mathrm{d}B(s) + \int_0^t (t-s)^{H-1/2}\,\mathrm{d}B(s)\right] \quad (1)$$

where $0 < H < 1$, $\Gamma(x)$ is the gamma function and $B(t)$ is BM. It has zero mean and covariance

$$\langle X_1(t)X_1(s)\rangle = \tfrac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}] \quad (2)$$

where the normalization $X_1(t) \to X_1(t)\big/\sqrt{\langle X_1^2(1)\rangle}$ is used. FBM is said to be H-self-similar

(H-ss) in the sense that

$$X_1(at) = a^H X_1(t) \tag{3}$$

where equality is in the finite joint distributions. The increments of FBM are stationary with

$$\langle (X_1(t+\tau) - X_1(t))^2 \rangle = \tau^{2H}. \tag{4}$$

FBM is the unique Gaussian self-similar process with stationary increments [10]. These two properties (i.e. self-similarity and stationary of increments) ensure a generalized spectral density for FBM

$$S_1(\omega) = \frac{c_H}{\omega^{2H+1}} \tag{5}$$

where c_H is an H-dependent constant.

The Lamperti transformation [1] provides one-to-one correspondence between a self-similar process $X(t)$ and a stationary process $Y(t)$ by the following time transformation

$$X(t) = t^H Y(b \ln t) \tag{6}$$

for $t > 0$, $b > 0$ and $X(0) = 0$. Conversely

$$Y(t) = \mathrm{e}^{-cHt} X(\mathrm{e}^{ct}) \tag{7}$$

for $t \in R$ and $c \in R$. The stationary process associated with FBM through the Lamperti transformation is given by

$$Y_1(t) = \frac{\mathrm{e}^{-2aHt}}{\sqrt{4aH}} X_1(\mathrm{e}^{2at}) \tag{8}$$

where the constants are chosen such that $Y_1(t)$ becomes the ordinary OU process when $H = 1/2$. This is a Gaussian process with zero mean and covariance

$$\begin{aligned} C_1(\tau) &= \frac{1}{8Ha}[\mathrm{e}^{2Ha\tau} + \mathrm{e}^{-2Ha\tau} - |\mathrm{e}^{a\tau} - \mathrm{e}^{-a\tau}|^{2H}] \\ &= \frac{1}{4Ha}[\cosh(2Ha\tau) - 2^{2H-1}(\sinh(a\tau))^{2H}]. \end{aligned} \tag{9}$$

By using the expansion $(1+x)^\alpha = 1 + \sum_{j=1}^\infty \binom{\alpha}{j} x^j$ with $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$, we obtain

$$\begin{aligned} C_1(\tau) &= \frac{1}{8Ha}[\mathrm{e}^{-2Ha\tau} + \mathrm{e}^{2Ha\tau}(1 - (1 - \mathrm{e}^{-2a\tau})^{2H})] \\ &= \frac{1}{8Ha}\left[\mathrm{e}^{-2Ha\tau} + \sum_{j=1}^{\infty}(-1)^{j+1}\binom{2H}{j}\mathrm{e}^{-2a(j-H)\tau}\right]. \end{aligned} \tag{10}$$

For $\tau \ll 1$,

$$\langle (Y_1(t+\tau) - Y_1(t))^2 \rangle \sim \tau^{2H} \tag{11}$$

which shows that $Y_1(t)$ is locally asymptotically stationary (or locally stationary for short). The large lag asymptotic behaviour of the covariance of $Y_1(t)$ can be obtained from (9). As $\tau \to \infty$, we obtain

$$\begin{aligned} C_1(\tau) &\approx \frac{1}{8Ha}[\mathrm{e}^{-2Ha\tau} - 2H\,\mathrm{e}^{-2(1-H)a\tau} + H(2H-1)\,\mathrm{e}^{-2(1-H)a\tau}] \\ &\sim \mathcal{O}\,\mathrm{e}^{-2a\tau(H\wedge(1-H))} \end{aligned} \tag{12}$$

where $x \wedge y$ denotes the minimum of (x, y). Thus, for large τ, $Y_1(t)$ is equivalent to the sum of a few mutually independent Gaussian Markov processes. The leading term of $C_1(\tau)$ for $\tau \to \infty$ is $C_1(\tau) \sim \mathrm{e}^{-2Ha\tau}$ for $H < 1/2$, and $C_1(\tau) \sim \mathrm{e}^{-2(1-H)a\tau}$ for $H > 1/2$.

The Fourier transform of equation (10) gives the spectral density of $Y_1(t)$ as

$$S_1(\omega) = \int_{-\infty}^{\infty} C_1(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \frac{1}{2}\left[\frac{1}{(2Ha)^2+\omega^2} + \frac{1}{H}\sum_{j=1}^{\infty}(-1)^{j+1}\binom{2H}{j}\frac{(j-H)}{(2a(j-H))^2+\omega^2}\right] \tag{13}$$

which reduces to $(a^2+\omega^2)^{-1}$ for the OU process when $H = 1/2$. By applying the Laplace transform to equation (9) it can be shown that the spectral density of $Y_1(t)$ can be expressed in the following form [11]:

$$S_1(\omega) = c_1(H,a)\frac{\left|\Gamma\left(1-H+\frac{\mathrm{i}\omega}{2a}\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}+\mathrm{i}\frac{\omega}{2a}\right)\right|^2\left(H^2+\frac{\omega^2}{4a^2}\right)} \tag{14}$$

where $c_1(H,a)$ is a constant that depends on H and a. With the help of the following asymptotic limit for the gamma function (see [12], p 760),

$$|\Gamma(x+\mathrm{i}|y|)| = \sqrt{2\pi}|y|^{x-1/2}\,\mathrm{e}^{-\frac{\pi|y|}{2}}\left[1+\mathcal{O}\left(\frac{1}{y}\right)\right] \tag{15}$$

we obtain the asymptotic limit of $S_1(\omega)$ as

$$S_1(\omega) \sim |\omega|^{-(2H+1)} \qquad |\omega| \to \infty. \tag{16}$$

We can also consider the 'finite memory' part of standard FBM, namely

$$X_1^+(t) = \sqrt{2H}\int_0^t (t-u)^{H-1/2}\,\mathrm{d}B(u) \tag{17}$$

where the normalization factor $\sqrt{2H}$ is used so that the variance is $|t|^{2H}$. $X_H^+(t)$ is also known as FBM of Riemann–Liouville type (RL-FBM) [13]. It has zero mean and a rather complicated covariance,

$$\left\langle X_1^+(s)X_1^+(t)\right\rangle = \frac{4Hs^{H+1/2}t^{H-1/2}}{(2H+1)}{}_2F_1\left(1/2-H, 1, H+3/2, \frac{s}{t}\right) \tag{18}$$

for $t > s > 0$, where ${}_2F_1$ is the Gauss hypergeometric function.

X_1^+ is a self-similar process like X_1, but its increments are non-stationary. Just as the case of standard FBM, Lamperti transformation of RL-FBM gives a stationary process $Y_1^+(t) = \mathrm{e}^{-2Hat}X_1^+(\mathrm{e}^{2at})$ with zero mean and covariance

$$C_1^+(\tau) = \left\langle Y_1^+(t)Y_1^+(t+\tau)\right\rangle = \frac{\mathrm{e}^{-a\tau}}{2a(H+1/2)}{}_2F_1(1/2-H, 1, H+3/2, \mathrm{e}^{-2a\tau}). \tag{19}$$

The large τ asymptotic of $C_\tau^+(\tau)$ can be obtained by using the series expansion of ${}_2F_1(\alpha,\beta,\gamma,z)$ [14],

$${}_2F_1(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+j)\Gamma(\beta+j)}{\Gamma(\gamma+j)}\frac{z^j}{j!} \tag{20}$$

which gives for $\tau \gg 1$,

$$C_1^+(\tau) \approx \frac{\mathrm{e}^{-a\tau}}{2a(H+1/2)}\left[1+\frac{\Gamma(3/2+H)\Gamma(3/2-H)}{\Gamma(1/2-H)\Gamma(5/2+H)}\,\mathrm{e}^{-2a\tau}\right]. \tag{21}$$

Similar to the case of standard FBM, in the large τ limit, RL-FBM behaves like a sum of a few independent OU processes.

The spectral density of $Y_1^+(t)$ can be determined by applying Fourier transform to $C_1^+(\tau)$, which gives

$$S_1^+(\omega) = \frac{\Gamma(3/2+H)}{2a(H+1/2)\Gamma(1/2-H)} \sum_{j=0}^{\infty} \frac{\Gamma(1/2-H+j)}{\Gamma(3/2+H+j)} \frac{2(1+2j)a}{((1+2j)a)^2+\omega^2}$$
$$= \frac{1}{(H+1/2)(\omega^2+a^2)} + \frac{(\Gamma(1/2+H))^2\cos(\pi H)}{2a\pi} \times \sum_{j=1}^{\infty} \frac{\Gamma(1/2-H+j)}{\Gamma(3/2+H+j)} \frac{2(1+2j)a}{((1+2j)a)^2+\omega^2}. \tag{22}$$

We can also show that the spectral density $S_1^+(\omega)$ approaches $\omega^{-(2H+1)}$ in the limit $\omega \to \infty$. Once again, by following the argument of Nuzman and Poor [11], we can show that the spectral density of $Y_1^+(t)$ can be expressed in the following compact form,

$$S_1^+(\omega) = c_1^+(a,H)\frac{|\Gamma(1/2+\mathrm{i}w/(2a)|^2}{|\Gamma(1+H+\mathrm{i}\omega)|^2} \tag{23}$$

where $c_1^+(a,H)$ is a constant. Again, by using (15), we obtain $S_1^+(\omega) \sim \omega^{-(2H+1)}$ for $\omega \to \infty$.

Note that the Lamperti transformation of two versions of FBM results in two stationary processes which have more properties in common as compared to their self-similar counterparts. A heuristic explanation is that the stationary processes Y_1 and Y_1^+ can be viewed roughly as the corresponding FBMs X_1 and X_1^+ on the logarithmic timescale. The compression of the clock time t to logarithmic time transforms the memory of 'infinite past' ($t = -\infty$ to $t = 0$) of standard FBM to a finite past ($\mathrm{e}^{-\infty} = 0$ to $\mathrm{e}^0 = 1$), whereas RL-FBM begins at $\mathrm{e}^0 = 1$ in logarithmic time. Therefore, we may say that the two stationary processes Y_1 and Y_1^+ started off at about the same time on the logarithmic scale, hence the similarities in their properties.

2.2. Process with stretched exponential covariance

Another Gaussian stationary process which can be regarded as a generalized OU process is one with mean zero and covariance given by the stretched exponential function,

$$C_2(\tau) = \langle Y_2(t+\tau)Y_2(t)\rangle = A\,\mathrm{e}^{-a|\tau|^\alpha} \tag{24}$$

where $0 < \alpha \leqslant 2$, and $A > 0, \alpha > 0$. Note that for $\alpha = 1, A = (2a)^{-1}$, we recover the covariance of OU processes from (24). For a discussion on the positive definiteness of equation (24), refer to [15]. Note that the stretched exponential function is also associated with the Levy stable process as it is identical to the characteristic function of the symmetric stable distribution [10]. For $\tau \ll 1$, the increments of $Y_2(t)$ satisfy

$$\langle (Y_2(t+\tau) - Y_2(t))^2\rangle = A\tau^{2H} \tag{25}$$

which shows that the increments of $Y_2(t)$ are locally stationary.

The Gaussian process with covariance (24) has been used to describe the velocity process of a particle inside a Sinai billiard with infinite horizon [16]. It is also used in nonlinear stochastic field theories [17], collective motion of worm-like micellar systems [18], and in geoscience such as the geopotential height of isobaric surfaces [19], etc. The stretched exponential function is also widely used to describe relaxation phenomena in disordered systems where the pertinent time-displaced correlation function decays as a stretched exponent.

Examples include oxide and polymeric glasses, spin correlations in Cu–Mn and Ag–Mn spin glasses, dielectric relaxation in a charge–density–wave system and neutron spin–echo measurements of ionic glasses, etc [20]. Although the exponential function is often used as a convenient fitting function for empirical data, a theoretical understanding of its underlying microscopic origin is still lacking.

The spectral density for the process with stretched exponential covariance can be expressed in a closed form only for $\alpha = 1$ and $\alpha = 2$. There are few cases for which the spectral densities can be obtained in terms of some higher mathematical functions [21]. For example, spectral densities for $\alpha = 1/2$ and $\alpha = 2/3$ are, respectively

$$S_2(\omega) = \sqrt{\frac{\pi}{2}}\frac{Aa}{|\omega|^{3/2}}\left[\cos\left(\frac{a^2}{4|\omega|}\right)\left(1-2C\left(\frac{a}{2\sqrt{|\omega|}}\right)\right)+\sin\left(\frac{a^2}{4|\omega|}\right)\left(1-2S\left(\frac{a}{2\sqrt{|\omega|}}\right)\right)\right] \tag{26}$$

and

$$S_2(\omega) = \sqrt{\frac{\pi}{3}}\frac{A}{|\omega|}\,\mathrm{e}^{\frac{2a^3}{27\omega^2}}\,W_{-\frac{1}{2},\frac{1}{6}}\left(\frac{4a^3}{27\omega^2}\right) \tag{27}$$

where $C(z)$ and $S(z)$ are the two Fresnel integrals and $W_{\mu,\nu}(z)$ is the Whittaker function. Note that for $\alpha = 2/3$ we have used the corrected version given by Garoni and Frankel [22], who have recently obtained expressions of $S_2(\omega)$ for rational values of α in terms of Meijer G functions and generalized hypergeometric functions.

The asymptotic behaviour of $S_2(\omega)$ can be obtained by a series expansion for large arguments ($|\omega| \gg 1$):

$$S_2(\omega) = -\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(\alpha j+1)}{|\omega|^{\alpha j+1}}\sin\left(\frac{j\alpha\pi}{2}\right) + \mathcal{O}(\omega^{-\alpha(n+1)+1}). \tag{28}$$

Thus, the asymptotic approximation of the spectral density of a process with stretched exponential covariance is

$$S_2(\omega) \sim \frac{\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi|\omega|^{\alpha+1}} \sim |\omega|^{-(\alpha+1)} \tag{29}$$

where $A = a = 1$ is used. For $\alpha = 2H$, $S_2(\omega)$ has the same asymptotic limit as $S_1(\omega)$.

2.3. Generalized OU process from fractional Langevin equation

Recall that the ordinary OU process is the stationary solution of the Langevin equation

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} + aY(t) = \eta(t) \qquad a > 0 \tag{30}$$

where $\eta(t)$ is the standard white noise with

$$\langle\eta(t)\rangle = 0 \qquad \langle\eta(t)\eta(s)\rangle = \delta(t-s). \tag{31}$$

The stationary solution is given by

$$Y(t) = \int_{-\infty}^{t}\mathrm{e}^{-a(t-u)}\eta(u)\,\mathrm{d}u. \tag{32}$$

Now we want to generalize equation (30) to a fractional Langevin equation such that its stationary solution can be regarded as a fractional OU process. There are two ways to extend equation (30) to a fractional Langevin equation. One direct way is the following

$$\frac{\mathrm{d}^\beta Y(t)}{\mathrm{d}t^\beta} + aY(t) = \eta(t) \qquad \beta > 0 \tag{33}$$

where the fractional derivative can be defined in term of the fractional integral [23]

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u \qquad \alpha > 0. \tag{34} $$

For $\gamma = -\alpha > 0$, the fractional derivative ${}_aD_t^{\gamma}$ is then defined as fractional integral of order $n-\gamma$ (with $n-1 < \gamma < n$) and the ordinary derivative of order n:

$$ {}_aD_t^{\gamma} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n {}_aD_t^{\gamma-n} f(t). \tag{35} $$

Equations (34) and (35) are known as the fractional integral and fractional derivative of RL type if $a = 0$ and the Weyl fractional integral and derivative for $a = -\infty$. We consider the case of the RL definition for equation (33).

For $n-1 < \nu < n$ and the following boundary conditions

$$ \left.\frac{\mathrm{d}^j Y(t)}{\mathrm{d}t^j}\right|_{t=0} = Y_j^0 \qquad j = 1, 2, \ldots, n-1 \tag{36} $$

the Laplace transform of equation (33) is

$$ s^{\beta}\widetilde{Y}(s) + a\widetilde{Y}(s) = \widetilde{\eta}(s) + \sum_{j=1}^{n} s^{\beta-j} Y_j^0 \tag{37} $$

which gives

$$ \widetilde{Y}(s) = \frac{\widetilde{n}(s)}{s^{\beta}+a} + \sum_{j=1}^{n} Y_{j-1}^0 \frac{s^{\beta-j}}{s^{\beta}+a}. \tag{38} $$

The inverse Laplace transform of equation (38) gives

$$ Y_3(t) = \sum_{j=1}^{n} Y_{j-1}^0 t^{j-1} E_{\beta,j}(-at^{\beta}) + \int_0^t (t-u)^{\beta-1} E_{\beta,\beta}(-a(t-u)^{\beta})\eta(u)\,\mathrm{d}u \tag{39} $$

where $E_{\alpha,\beta}$ is the generalized Mittag–Leffler function defined by [24]:

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)} \qquad \alpha > 0 \quad \beta > 0. \tag{40} $$

We assume all Y_j^0 equal to zero which has no effect on the subsequent conclusion to be drawn. $Y_3(t)$ is a Gaussian process with zero mean and the following covariance

$$ \begin{aligned} \langle Y_3(s)Y_3(t)\rangle &= \int_0^s \frac{E_{\beta,\beta}(-a(s-u)^{\beta})E_{\beta,\beta}(-a(t-u)^{\beta})}{(s-u)^{1-\beta}(t-u)^{1-\beta}}\,\mathrm{d}u \\ &= \sum_{j,k=0}^{\infty} \frac{(-a)^{j+k}}{\Gamma(\beta j+\beta)\Gamma(\beta k+\beta)} \int_0^s (s-u)^{\beta(j+1)-1}(t-u)^{\beta(k+1)-1}\,\mathrm{d}u \\ &= \sum_{j,k=1}^{\infty} \frac{(-a)^{j+k-2}}{\Gamma(\beta j+1)\Gamma(\beta k)} s^{\beta j} t^{\beta k-1} {}_2F_1\left(1, 1-\beta k, 1+\beta j, \frac{s}{t}\right) \end{aligned} \tag{41} $$

which shows that $Y_3(t)$ is a non-stationary process, hence it cannot be regarded as a generalized OU process which is supposed to be stationary. We can hope to obtain a stationary solution analogous to the stationary solution of the Langevin equation (32) by considering the solution of equation (33) as

$$ Y_3(t) = \int_{-\infty}^t (t-u)^{\beta-1} E_{\beta,\beta}(-a(t-u)^{\beta})\eta(u)\,\mathrm{d}u. \tag{42} $$

However, the integral (42) is divergent. Thus it is not possible to obtain a stationary process from the fractional Langevin equation (33).

On the other hand, it is possible to 'fractionalize' the Langevin equation in another way, namely

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+a\right)^{\beta} Y(t)=\eta(t) \qquad \beta>0. \tag{43}$$

Equation (43) can be solved using Fourier transform which gives

$$(a-\mathrm{i}\omega)^{\beta}\widetilde{Y}(\omega)=\widetilde{\eta}(\omega). \tag{44}$$

The Green function is

$$G(t)=\begin{cases}\dfrac{t^{\beta-1}\,\mathrm{e}^{-at}}{\Gamma(\beta)} & \text{for} \quad t>0\\ 0 & \text{for} \quad t<0.\end{cases} \tag{45}$$

The solution to equation (43) is

$$Y_3(t)=c_3'(a,\beta)\int_{-\infty}^{\infty}G(t-u)\eta(u)\,\mathrm{d}u \tag{46}$$

where $c_3'(a,\beta)$ is a constant which is chosen such that we recover the OU process for $\beta=1$. The covariance of $Y_3(t)$ is given by

$$C_3(\tau)\equiv\langle Y_3(t+\tau)Y_3(t)\rangle=\frac{a^{-2\nu}}{2^{\nu}\sqrt{\pi}\,\Gamma(\nu+1/2)}|a\tau|^{\nu}K_{\nu}(|a\tau|) \tag{47}$$

where $\nu=\beta-1/2$. K_ν is the modified Bessel function of the second kind, which can be expressed in terms of the modified Bessel function of the first kind $I_{\pm\nu}$:

$$K_{\nu}(z)=\frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(z)-I_{\nu}(z)]. \tag{48}$$

Note that

$$I_{\nu}(z)\sim\left(\frac{z}{2}\right)^{\nu}\frac{1}{\Gamma(\nu+1)} \qquad z\to 0 \tag{49}$$

which gives

$$K_{\nu}(z)\sim 2^{\nu-1}\Gamma(\nu)z^{-\nu} \qquad z\to 0. \tag{50}$$

From equation (50), we can compute the variance of $Y_3(t)$ which is

$$C_3(0)=2^{\nu-1}c_3(a,\nu)\Gamma(H) \tag{51}$$

where $c_3(a,\nu)=(2^{\nu}\sqrt{\pi}\,\Gamma(\nu+1/2)a^{2\nu})^{-1}$.

Just like $Y_2(t)$, the increments of $Y_3(t)$ satisfy the locally stationary property. It can be shown that

$$\langle(Y_3(t+\tau)-Y_3(t))^2\rangle\approx\frac{c_3(a,\nu)}{2^{\nu}\Gamma(\nu+1)\sin(\nu\pi)}|a\tau|^{2\nu} \tag{52}$$

which has the same form as for $Y_2(t)$ if we identify ν with H. We use $\nu=H$ for the subsequent discussion, and call $Y_3(t)$ the K-Bessel process.

The spectral density of $Y_3(t)$ is given by

$$S_3(\omega)=\frac{1}{(a^2+\omega^2)^{H+1/2}}. \tag{53}$$

Just as the OU process is known as the oscillator process with 'propagator' $(\omega^2+a^2)^{-1}$, the K-Bessel process can be called the fractional oscillator process with propagator $(\omega^2+a^2)^{-\beta}$. In the high-frequency limit $\omega\gg a$, $S_3(\omega)\sim\omega^{-(2H+1)}$, which is the generalized spectral density

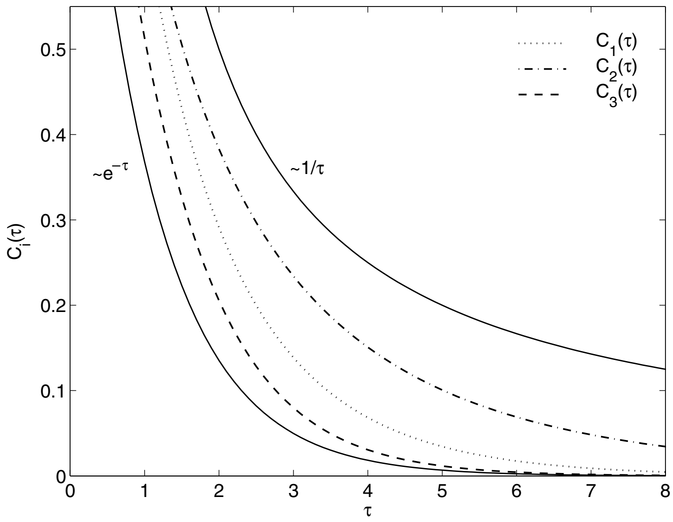

Figure 1. The correlation functions of the generalized OU processes.

of FBM. The form of its spectral density allows $Y_3(t)$ to be regarded as the 'correct' stationary analogue of FBM, similar to the relation between the OU process and BM. The simplicity of the spectral density of the *K*-Bessel process (or its higher-dimensional generalization *K*-Bessel field) has provided flexibility in its applications in modelling rainfall variability in hydrology [25], theory of turbulence [26], electrical noise model in semiconductors [27], spatial analysis in geoscience [28], etc. So far all the applications of the *K*-Bessel process are based on either its covariance (47) or spectral density (53). Our 'derivation' of this process based on the fractional Langevin equation (43) can be regarded as a useful input as far as simulations and modelling are concerned. As a side remark, we note that just as the stretched exponential function is better known as the characteristic function of the symmetric Levy stable distribution, the *K*-Bessel covariance function (47) has the same functional form as the characteristic function of the Pearson's distribution of the seventh kind [29].

The three generalized OU processes are non-Markovian processes with short-range dependence (SRD). The asymptotic behaviour of their covariances lies between the two extreme cases, namely the Markovian model with covariance $\sim e^{-a\tau}$ and the LRD model with covariance $\sim e^{-a\log\tau} = \tau^{-a}, 0 < a < 1$. For sufficiently large τ, covariances $C_i(\tau)$ of $Y_i(t), i = 1, 2, 3$, fall off slower or at most similar to that of the Markovian model, but they decay faster than that of the LRD model (see figure 1). However, there is one exception in the case of stretched exponential covariance when $1 < \alpha < 2$, i.e. the case of the super-exponential, which falls off faster than $e^{-a\tau}$.

The three generalized OU processes considered are all of short memory. However, there also exist Gaussian stationary processes which have the same local covariances as the generalized OU processes but have LRD. For example, the Gaussian stationary process with the power-law type of covariance $C(\tau) = (1 + |\tau|^{\alpha})^{\beta/\alpha}, 0 < \alpha \leqslant 2, \beta > 0$ is such a process. This non-self-similar Gaussian process has LRD for $0 < \beta < 1$ [30].

In the frequency domain, the spectral densities of generalized OU processes approach asymptotically to $\omega^{-(2H+1)}$ (for $\alpha = 2H, \nu = H$), the generalized spectral density of FBM (see figure 2). In the next section we see that this asymptotic behaviour of spectral densities is reflected in the local covariances of these processes.

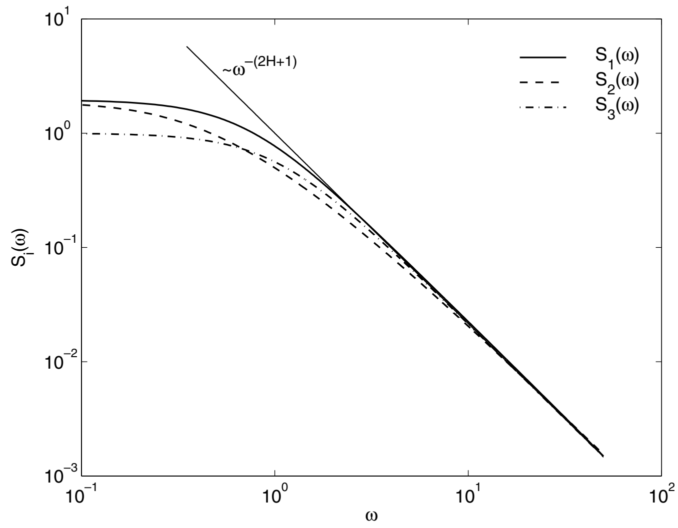

Figure 2. The spectral densities of the stationary processes $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ for $H = 0.75$ and $\alpha = 1.5$.

3. Local stationary representation of FBM

From the above discussion, it is clear that the three generalized OU processes have many properties in common. In particular, their spectral densities have the same large frequency limit, and they have local stationary increments. From the first property, we would then expect these processes to have similar covariances $C_i(\tau)$, $i = 1, 2, 3$ in the limit $\tau \ll 1$. This leads us to another way of linking the three generalized OU processes with FBM, namely they can be regarded as the locally stationary representations of FBM. In order to do this, it is necessary to introduce the notion of local stationarity as understood in geostatistics [8]. The term local stationary process usually denotes a non-stationary random process which is approximately stationary over regions which are sufficiently small. However, such a term has a slightly different meaning in geostatistical analysis. In geostatistics, a random process with stationary increments is known as an intrinsic random function (IRF) or to be more exact a zero-order IRF, and half the value of the variance of its increments is known as the variogram [8]. For example, FBM is an IRF with the variogram $|\tau|^{2H}/2$, $\tau \in R$.

Recall that a real function G is said to be positive-definite on R if for all $t_1, \ldots, t_n \in R$ and all real $\lambda_1, \ldots, \lambda_n$,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j G(t_j - t_i) \geqslant 0. \tag{54}$$

Positive-definiteness of G on a finite interval $I \subset R$ can also be defined in the same way with all t_i restricted to I. G is called a generalized covariance of an IRF $X(t)$ if

$$\left\langle \left(\sum_{i=1}^{n} \lambda_i X(t_i) \right)^2 \right\rangle = \sum_{i,j=1}^{n} \lambda_i \lambda_j G(t_j - t_i) \tag{55}$$

for all $\lambda_i \in R$ such that $\sum_{i=1}^{n} \lambda_i = 0$, and all $t_i \in R$. The linear combination $\sum \lambda_i X(t_i)$ is called an authorized linear combination [31] or equivalently an allowable linear combination (ALC) [8].

We can now define the notion of local stationarity used in geostatistics [8, 31]. Let K be a positive-definite function on a finite interval I. K is called a locally equivalent stationary covariance to a generalized covariance G for an IRF $X(t)$ in an interval I if

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j G(t_j - t_i) = \sum_{i,j=1}^{n} \lambda_i \lambda_j K(t_j - t_i) \tag{56}$$

for an ALC $\sum_{i=1}^{n} \lambda_i X(t_i)$, $t_1, \ldots, t_n \in I$. As the interval I is assumed to be bounded, the stationarity is then local in the sense that it holds on I. An IRF $X(t)$ is locally stationary over I if it has a representation $Y(t)$ that coincides on I with a stationary process $Y_k(t)$. In other words, for an ALC $\sum_{i=1}^{n} \lambda_i X(t_i)$,

$$\sum_{i=1}^{n} \lambda_i Y_k(t_i) = \sum_{i=1}^{n} \lambda_i X(t_i). \tag{57}$$

The stationary covariance $C(\tau)$ of $Y_k(t)$ is equivalent to the generalized covariance $G(\tau)$ on I:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j C(t_j - t_i) = \sum_{i,j=1}^{n} \lambda_i \lambda_j G(t_j - t_i). \tag{58}$$

We remark that there are other notions of local stationarity [32] and also that of local asymptotic stationarity [33].

Now let us consider FBM $X_1(t)$ as an example of an IRF and discuss its local stationary representations. Although IRFs such as FBM are non-stationary, some of them are essentially stationary on finite intervals. For an ALC of FBM, $\sum_{i=1}^{n} \lambda_i X_1(t_i)$ with $t_1, \ldots, t_n \in I$, we have

$$\left\langle \left(\sum_{i=1}^{n} \lambda_i X_1(t_i) \right)^2 \right\rangle = - \sum_{i,j=1}^{n} \lambda_i \lambda_j |t_j - t_i|^{2H}. \tag{59}$$

Note that the generalized covariance $G(\tau) = -|\tau|^{2H}$ is unique up to an additive constant. Matheron [34] has shown that an IRF $X(t)$ with the variogram $b|\tau|^{\alpha}$ has local equivalent stationary representation on $[-T, T]$ of the form

$$C(\tau) = b(A - |\tau|^{\alpha}) \qquad |\tau| \leqslant 2T \tag{60}$$

with

$$A \geqslant A_{\alpha} = \frac{T}{\sqrt{\pi}} \Gamma\left(\frac{1+\alpha}{2}\right) \Gamma(1 - \alpha/2). \tag{61}$$

The condition on A is to ensure the positive-definiteness of the covariance (60) so that it defines a covariance. By taking $b = 1, \alpha = 2H$, we obtain a local stationary representation of FBM on $[-T, T]$.

We next show that the three generalized OU processes can be regarded as the local stationary representation of FBM. First consider $Y_1(t)$ and for $|a\tau| \ll 1$, its covariance (10) becomes

$$C_1(\tau) = \frac{1}{4Ha}[1 - |a\tau|^{2H}] + \mathcal{O}(|a\tau|^2). \tag{62}$$

Up to $\mathcal{O}(|a\tau|^2)$, $C_1(\tau)$ is of the form (60) and $Y_1(t)$ is a local stationary representation of FBM on $[-a\tau, a\tau]$. By adjusting the positive parameter a, we have some flexibility in the choice of the interval of the local stationarity.

As for the stretched exponential process $Y_2(t)$, the local covariance is given by

$$C_2(\tau) = A[1 - |a\tau|^{2H} + \mathcal{O}(|a\tau|^{4H})] \tag{63}$$

for $|a\tau| \ll 1$. Again, $Y_2(t)$ can be regarded as a local stationary representation of FBM.

Finally, using equations (47), (48) and (49) the local covariance of the generalized OU process $Y_3(t)$ obtained from the fractional Langevin equation can be shown to be

$$C_3(\tau) = \frac{c_3(a,H)\Gamma(H)}{2^{1-H}}\left[1 - \frac{\pi}{2^{2H}\Gamma(H+1)\sin(\pi H)}|a\tau|^{2H}\right] + \mathcal{O}(|a\tau|^2) \tag{64}$$

for $|a\tau| \ll 1$, and $\nu = H$. Note that expression (48) for $K_H(|a\tau|)$ has been used to obtain (64). Again, equation (64) is a local equivalent stationary covariance for FBM up to $\mathcal{O}(|a\tau|^2)$.

The three generalized OU processes can be considered as the local stationary representation of FBM. Thus, at the cost of a relatively minor operating restriction (namely the need to use the allowable linear combination), we gain the possibility of using local stationary Gaussian processes to represent IRFs such as FBM. The local stationary covariances of the three generalized OU processes are of the form

$$C_i(\tau) = C_i(0) - A|\tau|^\alpha(1 + \mathcal{O}(|\tau|^\beta)) \qquad \tau \to 0 \tag{65}$$

for some $\beta > 0$. For $Y_1(t)$ and $Y_3(t)$, $\beta = 2 - 2H$, whereas for $Y_2(t)$, $\beta = 2H$. On the other hand, not every IRF has a local stationary representation. One such example is the IRF with $\gamma(\tau) = \tau^2$. There does exist a real number $C(0)$ such that $C(\tau) = C(0) - \tau^2$ is a covariance function defined on a neighbourhood of $\tau = 0$. We remark that such differences in β would result in different contributions to the bias in the estimation of the parameter H in applications [35].

A stationary process with covariance $C(\tau)$ of the form (65) for $\tau \to 0$ is said to be locally self-similar [35]. For Gaussian processes, this notion of local self-similarity is equivalent to the local asymptotic self-similarity introduced by Benassi *et al* [36] to describe multi-fractional Brownian motion (MBM) which is FBM indexed by a variable Hurst exponent $H(t)$. MBM $X(t)$ is locally asymptotically self-similar if

$$\lim_{\rho\to 0^+}\left[\frac{X(t+\rho u) - X(t)}{\rho^{H(t)}}\right] = X_{H(t)}(u) \tag{66}$$

where $X_{H(t)}(u)$ is FBM indexed by $H(t)$ and the equality is in the sense of distributions. Local asymptotic stationarity of Y_i implies $Y_i(t+\rho u) - Y_i(t) = Y_i(t)$ for $\rho \to 0^+$, which together with equation (65) can be shown to satisfy equation (66) if $H(t)$ is replaced by H. We can view a local asymptotic self-similar (or local self-similar) Gaussian process at a point t_o as the tangent process at t_o that tends in distribution to a self-similar process. We note that for a stationary Gaussian process with covariance (65), its fractal dimension is shown to be given by $2 - \alpha/2$ [37, 38].

Local equivalent stationary covariances have applications in computation and simulation in addition to being of theoretical interest. They allow optimal prediction and also facilitate fast and exact simulation of IRFs [8, 28, 39].

4. Self-similar processes from generalized OU processes

In this section, we consider self-similar processes obtained from the generalized OU processes through the Lamperti transformation. Since one of the self-similar processes, namely FBM associated with $Y_1(t)$, is well-studied, our discussion mainly focuses on the self-similar processes obtained by applying Lamperti transformations to the generalized OU processes $Y_2(t)$ and $Y_3(t)$.

First we briefly describe the characterization of LRD or long memory since processes with LRD are intimately related to self-similar processes. An intuitive way of looking at the property of LRD is that events that are arbitrarily distant (either in time or in space) may still influence one another. There are several ways of characterizing LRD [40]. The usual way of defining LRD in the time domain is in terms of decay rates of long-lag covariances, or in the frequency domain in terms of rates of divergence of spectral densities at low frequencies. Consider a stationary finite variance process $X(t)$ with covariance $C(k) = \langle X(j)X(j+k)\rangle$. A standard way to define LRD is based on the divergence of the sum of the covariance series:

$$\sum_{k=-\infty}^{\infty} C(k) = \infty. \tag{67}$$

The slow rate of decay of the covariance of the process is an indicator of LRD. The process is called SRD or short memory if the summation (67) is finite.

Equivalently, LRD can also be characterized in terms of spectral density. The process X has LRD if its spectral density $S(\omega)$ satisfies

$$S(\omega) \sim c|\omega|^{-\gamma} \qquad \omega \to 0 \tag{68}$$

where $0 < \gamma < 1$ and $c > 0$ is a constant. $S(\omega)$ diverges when the frequencies tend to zero. Equation (68) implies

$$C(k) \sim 2c\Gamma(1-\gamma)\sin\left(\frac{\pi\gamma}{2}\right)k^{\gamma-1} \qquad k \to \infty \tag{69}$$

i.e. the covariance function decays hyperbolically, which is the cause for the divergence of the sum (67). On the other hand, SRD is characterized by a spectral density, which is bounded at low frequencies, and a covariance function that decreases faster than equation (69), or exponentially fast in the case of the Markov process.

For the consideration of the LRD of non-stationary Gaussian processes, the following characterization of LRD can be used [41, 42]. Let $\widehat{C} = C(\tau)/C(0)$ denote the normalized covariance of $Y(t)$, a Gaussian stationary process. The process $Y(t)$ is said to have LRD if

$$\int_0^{\infty} |\widehat{C}(\tau)|\,\mathrm{d}\tau = \infty. \tag{70}$$

If the integral (70) is finite, $Y(t)$ has SRD.

The condition (70) can be generalized to the non-stationary Gaussian process. In this case, it is necessary to consider the correlation function

$$R(s,t) = \frac{C(s,t)}{[C(s,s)C(t,t)]^{1/2}} \tag{71}$$

instead of the covariance. A non-stationary Gaussian process $X(t)$ is said to have LRD if

$$\int_0^{\infty} |R(t,t+\tau)|\,\mathrm{d}\tau = \infty. \tag{72}$$

Alternatively, $Y(t)$ is said to have LRD if for $\tau \to \infty$,

$$R(t,t+\tau) \sim \tau^{\gamma} \qquad \forall t > 0 \tag{73}$$

with $-1 < \gamma < 0$ [42]. We apply the above characterization of LRD (and SRD) to both generalized OU processes and their self-similar counterparts.

4.1. LRD characteristics of FBM

For completeness we briefly discuss the LRD of FBM. Nearly all the discussions of the LRD related to FBM are considered in terms of its increment process or the fractional Gaussian noise (FGN) which is stationary [10]. However, we can also consider the LRD of FBM directly by looking at its correlation function.

For fixed $s \geqslant 0$ and $t \to \infty$, the Taylor expansion of the covariance of FBM gives the leading term as

$$\langle X_1(s)X_1(t)\rangle \sim \begin{cases} t^{2H-1} & \text{for} \quad H > \frac{1}{2} \\ s^{2H} & \text{for} \quad H < \frac{1}{2}. \end{cases} \tag{74}$$

This leads to the large t asymptotic correlation for fixed s to behave as

$$\langle X_1(s)X_1(t)\rangle \sim \begin{cases} t^{H-1} & \text{for} \quad H > \frac{1}{2} \\ s^{-H} & \text{for} \quad H < \frac{1}{2} \end{cases} \tag{75}$$

when $t \to \infty$. Thus by using condition (73), we conclude that FBM has LRD for $H \in (0, 1)$ except for $H = 1/2$, which corresponds to BM. In the case of BM, we have trivially SRD with independent increments. On the other hand, the stationary counterpart of FBM, $Y_1(t)$, has SRD since equation (10) shows that its covariance $C_1(\tau)$ decreases exponentially for large τ.

Likewise, RL-FBM $X_1^+(t)$ also has LRD. The correlation of $X_1^+(t)$ can be expressed as

$$\begin{aligned} R_1'(t, t+\tau) &= \frac{\sqrt{2H}}{(H+1/2)}\sqrt{\frac{t}{t+\tau}}\,{}_2F_1\left(1/2-H, 1, 3/2+H, \frac{t}{t+\tau}\right) \\ &= \frac{\sqrt{2H}}{(H+1/2)}\frac{\Gamma(3/2+H)}{\Gamma(1/2-H)}\sqrt{\frac{t}{t+\tau}}\sum_{j=0}^{\infty}\frac{\Gamma(1/2-H+j)}{\Gamma(3/2+H+j)}\left(\frac{t}{t+\tau}\right)^j \end{aligned} \tag{76}$$

for $\tau > 0$. This is a monotonic decreasing series, thus it suffices to apply condition (72) to the leading term ($j = 0$):

$$\frac{\sqrt{2H}}{(2H+1/2)}\int_0^\infty (t+\tau)^{-1/2}\,\mathrm{d}\tau = \infty.$$

Since RL-FBM is defined for $H > 0$, hence it has LRD for all positive H, except $H = 1/2$. Again, the stationary counterpart of RL-FBM has SRD for all $H > 0$ since its covariance (21) decays exponentially. The sample paths of the stationary process $Y_1(t)$ and the H-ss process $X_1(t)$ are shown in figure 3; see the appendix for a brief discussion on the simulation techniques.

4.2. Self-similar process associated with the stretched exponential process

FBM provides a relatively simple model for scaling phenomena and fractal time series. However, it is unlikely that real-life phenomena can be described by a process which is characterized by just a single parameter, H. In the subsequent discussion, we use the Lamperti transformation to obtain two new Gaussian H-ss processes which are determined by more than one parameter.

First, we consider the stretched exponential process $Y_2(t)$ which has SRD since

$$\begin{aligned} \int_0^\infty C_2(\tau)\,\mathrm{d}\tau &= A\int_0^\infty \exp[-a|\tau|^\alpha]\,\mathrm{d}\tau \\ &= \frac{A}{a^{1/\alpha}}\frac{\Gamma(\alpha)}{\alpha} \qquad A, a > 0 \qquad 0 < \alpha \leqslant 2. \end{aligned} \tag{77}$$

Applying the Lamperti transformation

$$X(t) = t^H Y(a \ln t) \tag{78}$$

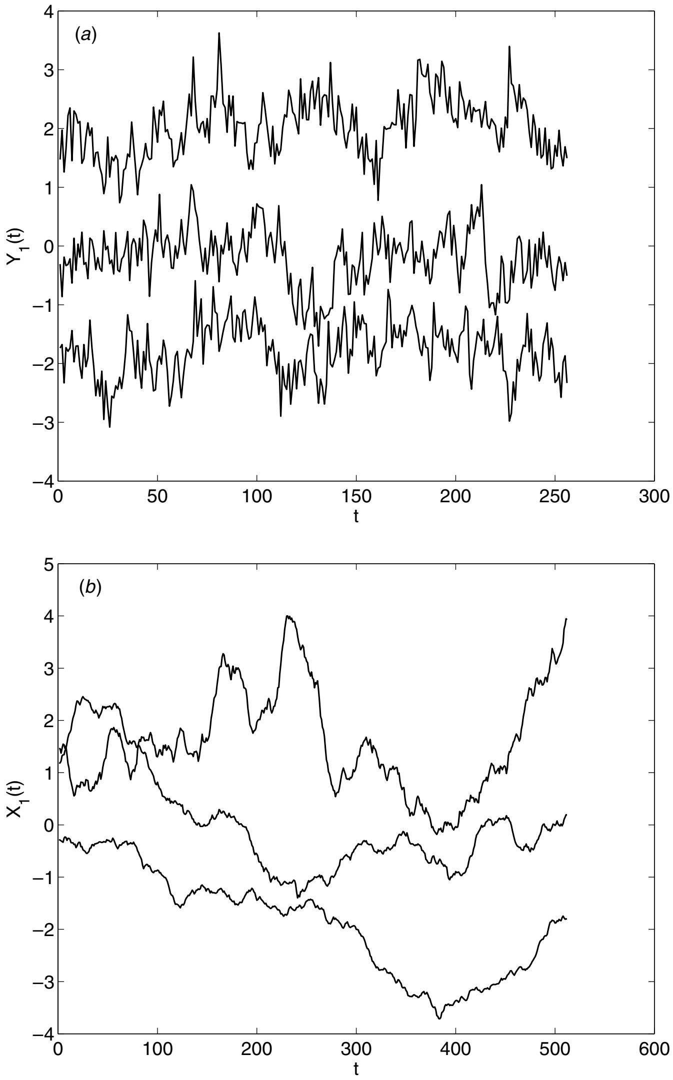

Figure 3. The sample paths of (*a*) the stationary process $Y_1(t)$ and (*b*) the self-similar process $X_1(t)$ for $H = 0.75$.

to $Y_2(t)$ results in a H-ss Gaussian process $X_2(t)$ with the following covariance

$$\langle X_2(s)X_2(t)\rangle = As^H t^H \exp\left[-a\left(\log\frac{t}{s}\right)^\alpha\right] \tag{79}$$

and variance At^{2H}. The increments of $X_2(t)$ are non-stationary. In fact, the only Gaussian self-similar process with stationary increments is FBM. However, the increments of $X_2(t)$ satisfy a weaker property of local stationarity for the special case with $a = 2H$. For this particular value of α, we have

$$\langle (X_2(t+\tau) - X_2(t))^2\rangle \sim \tau^{2H} + \mathcal{O}(\max(\tau^2 t^{2H-2}, \tau^{2H+1}t^{-1}) \tag{80}$$

where $0 < \tau \ll t$.

The correlation function is

$$R_2(t+\tau, t) = \exp\left[-a\left(\log\frac{t+\tau}{t}\right)^{\alpha}\right] \tag{81}$$

for $\tau \geqslant 0$. By a change of variable, we have

$$\int_0^\infty R_2(t+\tau, t)\,\mathrm{d}\tau = t\int_0^\infty \mathrm{e}^{u-au^{\alpha}}\,\mathrm{d}u \tag{82}$$

which diverges for $\alpha < 1$. Thus, by the Lamperti transformation of the process with stretched exponential covariance, we obtain a Gaussian H-ss process with LRD for $\alpha < 1$. Note that there are three parameters: the self-similar scaling exponent H, the stretched exponent α and the multiplicative constant a. Here the parameter a plays a secondary role of controlling the 'size' of short memory when $\alpha > 1$.

Note that we cannot use condition (82) to characterize LRD when $\alpha = 1$. In this case, the process $Y_2(t)$ reduces to the ordinary OU process, which is a stationary Gaussian Markov process. The Lamperti transformation of $Y_2(t)$ for $\alpha = 1$ gives a Gaussian self-similar process with mean zero and covariance

$$\langle X_2(s)X_2(t)\rangle = s^{H+a}t^{H-a} \qquad s < t. \tag{83}$$

Recall that a Gaussian process with covariance $C(t, s)$ is Markovian if

$$C(t,s) = \frac{C(t,u)C(u,s)}{C(u,u)} \qquad t > u > s \tag{84}$$

which can be easily verified for equation (83). This result can also be deduced directly from the fact that the Lamperti transformation preserves the Markov property [43, 44]. The uniqueness of the OU process as a stationary Gaussian Markov process implies that equation (83) defines an entire class of Gaussian H-ss Markov processes parametrized by H and a. For $H = a$ it reduces to the time-rescaled BM $B(t^{2H})$. The LRD condition (72) fails to apply in this case since the correlation associated with equation (83) satisfies this condition for $0 < a < 1$, contradicting the SRD which characterizes Markov processes.

The sample paths of the stationary process $Y_2(t)$ and the self-similar process $X_2(t)$ are shown in figure 4.

4.3. Self-similar process associated with the K-Bessel process

We first show that the stationary K-Bessel process has SRD. We apply the condition on the covariance:

$$\begin{aligned}\int_0^\infty C_3(\tau)\,\mathrm{d}\tau &= c_3(a,H)\int_0^\infty (a\tau)^H K_H(a\tau)\,\mathrm{d}\tau \\ &= \frac{c_3(a,H)2^{H-1}\Gamma(H+1/2)}{a}.\end{aligned} \tag{85}$$

Equation (85) shows that $Y_3(t)$ has SRD for all $H > 0, a > 0$.

The self-similar process $X_3(t)$ obtained from $Y_3(t)$ using the Lamperti transformation has zero mean and covariance

$$\langle X_3(t)X_3(s)\rangle = c_3(a,\nu)(ts)^H\left(a\ln\frac{t}{s}\right)^{\nu} K_\nu\left(a\ln\frac{t}{s}\right) \tag{86}$$

with $t > s$. By using equation (50) for the small z asymptotic of the modified Bessel function $K_H(z)$, we obtain the variance

$$\langle (X_3(t))^2\rangle = \frac{\Gamma(\nu)}{2^{1-H}}c_3(a,\nu)t^{2H}. \tag{87}$$

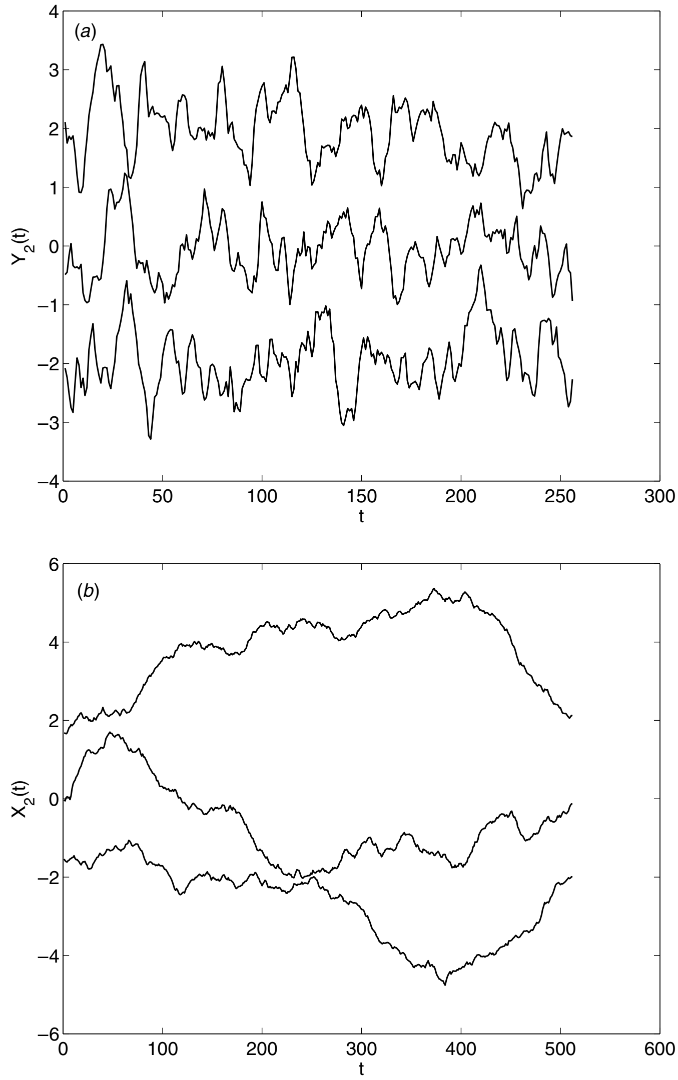

Figure 4. The sample paths of (a) the stationary process $Y_2(t)$ and (b) the self-similar process $X_2(t)$ for $\alpha = 1.5$.

The increments of $X_3(t)$ are non-stationary. They are locally stationary only if $\nu = H$, just as the case of $X_2(t)$. For $\tau \ll t$, using equations (86) and (87) and relations (48) and (49) for the modified Bessel function gives

$$\langle (X_3(t+\tau) - X_3(t))^2 \rangle \sim \tau^{2H} + \mathcal{O}(\max(\tau^2 t^{2H-2}, \tau^{2H+1} t^{-1})). \tag{88}$$

The correlation function is given by

$$R_3(t+\tau, t) = \left(a \ln\left(1 + \frac{\tau}{t}\right)\right)^{\nu} K_\nu \left(a \ln\left(1 + \frac{\tau}{t}\right)\right). \tag{89}$$

With the appropriate change of variable, we obtain

$$\int_0^\infty R_3(t+\tau, t)\, \mathrm{d}\tau = \frac{t}{a} \int_0^\infty \mathrm{e}^{\frac{u}{a}} u^\nu K_\nu(u)\, \mathrm{d}u. \tag{90}$$

By generalizing a result in [14, p 708, no 16], we obtain

$$I(z) = \int_0^z \mathrm{e}^{\frac{u}{a}} u^{\nu} K_{\nu}(u)\,\mathrm{d}u = \frac{az^{1+H}\,\mathrm{e}^{z}}{2H+1}(K_{\nu}(z) + K_{\nu+1}(z)) - \frac{2\Gamma(\nu+1)}{2\nu+1}. \tag{91}$$

For large z

$$K_{\nu}(z) \simeq \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left(1 + \frac{4\nu^2 - 1}{8z}\right). \tag{92}$$

Using equations (91) and (92), we obtain

$$\lim_{x\to\infty} I(x) = \lim_{x\to\infty} x^{\nu+\frac{1}{2}}\,\mathrm{e}^{(\frac{1}{a}-1)x} = \infty \qquad 0 < a < 1. \tag{93}$$

Thus the integral (90) is divergent for $0 < a < 1$, implying LRD for the process $X_3(t)$. As the parameter ν does not determine the memory range of $X_3(t)$, it can be identified with H to ensure its increment process satisfies the local stationary property. Thus, the self-similar Gaussian process $X_3(t)$ is a two-parameter process where H determines the self-similar scaling and parameter a controls the memory range of the process. The sample paths of $Y_3(t)$ and $X_3(t)$ are shown in figure 5.

5. Summary and concluding remarks

Three types of generalized OU process are considered and their properties studied. It is found that they have similar properties, in particular their local covariances have the same form: $C_i(\tau) \sim C_i(0) - |\tau|^{2H}$, $\tau \to 0$ (it is assumed that $\nu = H$ and $\alpha = 2H$ unless specified otherwise). This local behaviour in the time domain is reflected in the frequency domain where all three spectral densities have the same large frequency limit $|\omega|^{-(2H+1)}$, similar to the generalized spectral density of FBM. Such a local property also allows the generalized OU processes to provide the local equivalent representations of FBM. These local equivalent stationary representations of FBM allow the optimal prediction and fast accurate simulation of FBM and integrated FBM [39]. We remark that the local stationary representation only applies to IRFs such as FBM. The H-ss processes associated with the stretched exponential process and the K-Bessel process do not have stationary increments, hence the local stationary representation does not exist for these processes.

We have obtained two new H-ss processes (in addition to FBM) by applying the Lamperti transformation to the generalized OU processes. In contrast to FBM, the two new self-similar processes $X_2(t)$ and $X_3(t)$ are mathematically less tractable since they do not have stationary increments. However, with the identifications $\nu = H$ and $\alpha = 2H$, increments of $X_2(t)$ and $X_3(t)$ satisfy a weaker property of local stationarity. In the case of $\alpha \neq 2H$, $X_2(t)$ is characterized by three parameters H, α and a. H is the self-similar index, α plays the role of memory index with LRD for $0 < \alpha < 1$, and a determines the 'size' of SRD when $1 < \alpha \leqslant 2$. In the case of $X_3(t)$, we have a H-ss process parametrized by two (when $\nu = H$) parameters H and a. The process has LRD when $a < 1$. $X_2(t)$ and $X_3(t)$ provide more flexible models for scaling and LRD phenomena such as meteorological time series, financial time series, internet network traffic, DNA sequences, etc.

Finally we give some comments on the notion of LRD. The question concerning the actual meaning of LRD and its causes are still far from being resolved. In our discussion, we have used the definitions of LRD based on decay rates of the covariances and correlation functions. Doubts about the universal validity of such definitions have been raised and

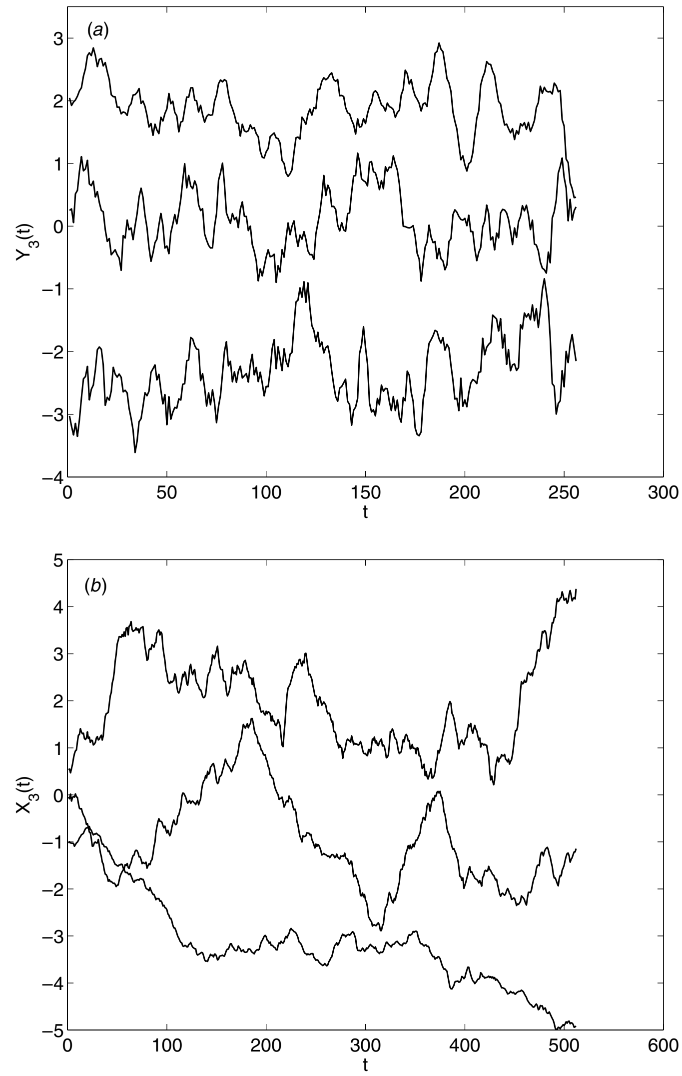

Figure 5. The sample paths of (*a*) the stationary process $Y_3(t)$ and (*b*) the self-similar process $X_3(t)$ for $\nu = 0.75$.

alternative approaches have been suggested [45]. In particular, the characterization of LRD for non-stationary processes and non-Gaussian processes is shown to be inadequate. In section 4, we have shown that the test for LRD based on correlation function is not applicable to an entire class of non-stationary *H*-ss Gaussian Markov processes.

Recently there have been debates on whether the apparent LRD behaviour in economic time series and internet traffic data is a statistical artefact originating from stochastic properties (such as non-stationarity effects, regime switching or structural change, etc) within a SRD model [46, 47]. LRD is considered for long enough time series, and its effect can disappear in shorter or intermediate time series. Some recent work indicated that non-Markovian SRD models with covariances of sub-exponential type such as $e^{-a\sqrt{\tau}}$ capture better the empirical data and provide good performance prediction for video traffic with a finite buffer [48]. Despite strong evidence for self-similar LRD models in internet traffic studies, there have been reports that suggest that certain generalized Markovian models are adequate for internet traffic even with a high value of scaling index H [49]. There is a growing consensus among physicists and engineers that a real-life time series cannot be completely described by a particular model alone. More studies are needed in order to provide a deeper understanding of LRD, its possible causes and consequences.

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Appendix

The simulation of stationary processes can be performed based on the spectral densities using the Fourier spectral technique. A simple algorithm is obtained by considering the discrete Fourier transform of a N-point discrete stationary sequences X, namely

$$X(t) = \sum_{k=0}^{N-1} a_k \, \mathrm{e}^{2\pi \mathrm{i} k t} \tag{A.1}$$

since the coefficients $\{a_k\}$, $k = 0, 1, \ldots, N-1$ are related to the spectral density $S(k)$ through $\langle |a_k|^2 \rangle \sim S(k)$. Readers are referred to [50] for the details of this standard technique.

The simulation of non-stationary Gaussian H-ss processes is rather difficult when their increments are non-stationary. In such cases, methods such as the mid-point displacement technique fail to give a satisfactory result. Moreover, a direct resampling technique based on the Lamperti transformation of the discrete sample set of the stationary process to produce the H-ss process has proven to be computationally extensive. Therefore, we resort to a well-known technique for simulating non-stationary Gaussian processes based on the Karhunan–Loeve (K–L) expansion, which is also known as the eigenfunction expansion [51].

Consider an orthogonal decomposition of a stochastic process $X(t)$, $t \in [0, T]$ written in the form

$$X(t) = \sum_{k=1}^{\infty} Z_k \phi_k(t) \qquad 0 < t < T \tag{A.2}$$

where $\{\phi_k(t)\}$ are the eigenfunctions of the covariance function $C(t_1, t_2)$ of the process $X(t)$ which is assumed to have zero mean. If the covariance kernel $C(t_1, t_2)$ is symmetric, i.e. $C(t_1, t_2) = C(t_2, t_1)$, then ϕ_k are the solutions of the Fredholm integral,

$$\int_0^T C(t_1, t_2) \phi(t_2) \, \mathrm{d}t_2 = \lambda \phi(t_1) \tag{A.3}$$

where λ denotes the eigenvalues $\{\lambda_k\}$, each of which corresponds to an eigenfunction $\phi_k(t)$. $\{Z_k\}$ is a set of orthogonal random variables all having zero mean, and the variance of each Z_k is given by λ_k such that

$$\langle Z_m Z_n \rangle = \delta_{mn} \lambda_n \tag{A.4}$$

$$\left\langle Z_k^2 \right\rangle = \lambda_k. \tag{A.5}$$

For each of the Gaussian H-ss processes considered here, we calculate the covariance matrix based on the analytical expression and the numerical eigenfunctions are determined. The sample paths are generated using the eigenfunction expansion (A2).

References

[1] Lamperti J W 1962 Semi-stable stochastic processes *Trans. Am. Math. Soc.* **104** 62–78

[2] Mandelbrot B B and van Ness J W 1968 Fractional Brownian motion, fractional noises and applications *SIAM Rev.* **10** 422–37

[3] Gouyet J F 1996 *Physics of Fractal Structures* (New York: Springer)

[4] Rothschild W G 1998 *Fractal in Chemistry* (New York: Wiley)

[5] Voss R 1992 Evolution of long-range fractal correlations and $1/f$ noise in DNA base sequences *Phys. Rev. Lett.* **68** 3805–8

[6] Mantegna R N and Stanley H E 2000 *An Introduction to Econophysics* (Cambridge: Cambridge University Press)

[7] Park K and Willinger W (eds) 2000 *Self-Similar Network Traffic and Performance Evaluation* (New York: Wiley)

[8] Chile J and Delfiner P 1999 *Geostatistics: Modeling Spatial Uncertainty* (New York: Wiley)

[9] Doukhan P, Oppenheim G and Taqqu M S 2002 *Theory and Applications of Long-Range Dependence* (Boston: Birkhauser)

[10] Samorodnitsky G and Taqqu M 1994 *Stable Non-Gaussian Random Processes* (New York: Chapman and Hall)

[11] Nuzman C and Poor H V 2000 Linear estimation of self-similar processes via Lamperti's transformation *J. Appl. Prob.* **37** 429–52

[12] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 *Integrals and Series* vol 3 (New York: Gordon and Breach) p 760

[13] Lim S C and Sithi V M 1995 Asymptotic properties of the fractional Brownian motion of Riemann–Liouville type *Phys. Lett.* A **206** 311–7

[14] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series, and Products* (San Diego, CA: Academic)

[15] Yaglom A M 1987 *Correlation Theory of Stationary and Related Random Functions* (New York: Springer)

[16] Bouchaud J P and Doursal P L 1985 Numerical study of a *d*-dimensional periodic Lorentz gas with universal properties *J. Stat. Phys.* **41** 225–48

[17] Schwartz M and Edwards S F 2002 Stretched exponential in non-linear stochastic field theories *Physica* A **312** 363–68

[18] Seto H, Kato T, Monkenbusch M, Takeda T, Kawabata Y, Nagao M, Okuhara D, Imai M and Komura S 1999 Collective motion of a network of wormlike micelles *J. Phys. Chem. Solids* **60** 1371–3

[19] Gandin L S 1963 *Objective Analysis of Meteorological Fields* (Jerusalem: Israel Programs for ScientificTranslation)

[20] Palmer R G, Stein D L, Abrahams E and Anderson P W 1984 Models of hierarchically constrained dynamics for glassy relaxation *Phys. Rev. Lett.* **53** 958–61

[21] Zolotarev V M 1986 *One Dimensional Stable Distributions* (Providence, RI: American Mathematical Society)

[22] Garoni T M and Frankel N E 2002 Levy flights: exact results and asymptotics beyond all orders *J. Math. Phys.* **43** 2670–89

[23] Samko S G, Kilbas A A and Maritchev D I 1993 *Integral and Derivatives of the Fractional Order and Some of Their Applications* (Amsterdam: Gordon and Breach)

[24] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1995 *Higher Transcendental Functions* vol 3 (New York: McGraw-Hill) ch 18

[25] Rodriguez-Iturbe I and Mejia J M 1974 The design of rainfall networks in time and space *Water Resour. Res.* **10** 713–28

[26] Monin A S and Yaglom A M 1975 *Statistical Fluid Mechanics* vol 2 (Cambridge, MA: MIT Press)

[27] Blanc-Lapierre A and Fortet R 1968 *Theory of Random Functions* vol II (New York: Gordon and Breach)

[28] Stein M L 1999 *Interpolation of Spatial Data: Some Theory of Kriging* (New York: Springer)

[29] Ushakov N G 1999 *Selected Topics in Characteristic Functions* (Zeist: VSP)

[30] Gneiting T J 2000 Power-law correlations related models for long-range dependence and their simulation *Appl. Prob.* **37** 1104–9

[31] Matheron G 1973 The intrinsic functions and their applications *Adv. Appl. Prob.* **5** 439–68

[32] Mallat S, Papanicolaou G and Zhang Z 1998 Adaptive covariance estimation and locally stationary processes *Ann. Statist.* **26** 1–47

[33] Lim S C 2001 Fractional Brownian motion and multifractional Brownian motion of Riemann–Liouville type *J. Phys. A: Math. Gen.* **34** 1301–10

[34] Matheron G 1974 The theory of regionalized variables and its applications *Technical Report: Note Geostatisque* vol 125 (Paris: Centre de Geostatistique, Ecole des Mines de Paris)

[35] Kent J T and Wood A T A 1997 Estimating the fractal dimension of locally self-similar Gaussian process by using increments *J. R. Statist. Soc.* B **59** 679–99

[36] Benassi A, Jaffard S and Roux D 1997 Elliptic Gaussian random processes *Rev. Math. Iberoamericana* **13** 19–90
[37] Adler R J 1981 *The Geometry of Random Fields* (New York: Wiley)
[38] Taylor C C and Taylor S J 1991 Estimating the dimension of fractal *J. R. Statist. Soc.* B **53** 353–64
[39] Stein M L 2002 Fast and exact simulation of fractional Brownian surface *J. Comput. Graphical Statist.* at press
[40] Beran J 1994 *Statistics for Long-Memory Processes* (New York: Chapman and Hall)
[41] Flandrin P, Borgnat P and Amblard P O 2002 From stationary to self-similarity, and back: variations on the Lamperti transformation *Preprint TS-0201 ENS Lyon*
[42] Ayache A, Cohen S and Levy-Vehel J 2000 The covariance structure of multifractional Brownian motion, with application to long-range dependence *Proc. IEEE Int. Conf. on Acoustic, Speech and Signal ICASSP'00 (Istanbul)*
[43] Adler R J, Cambanis S and Samorodnitsky G 1990 On stable Markov processes *Stoch. Process Appl.* **34** 1–17
[44] Burnecki K, Maejima M and Weron A 1997 The Lamperti transformations of self-similar processes *Yokohama Math. J.* **44** 25–41
[45] Rachev S T and Samorodnitsky G 2001 Long strange segments in a long-range dependent moving average *Stoch. Process. Appl.* **93** 119–48
[46] Mikosh T and Starica D 2000 Is it really long memory we see in financial returns? *Extremes and Integrated Risk Management* ed P Embrechts (London: Risk Books UBS Warburg) p 149
[47] Cao J, Cleveland S W, Lin D and Sun D X 2001 On the non-stationarity of internet traffic *Proc. ACM SIGMETRICS'01* pp 102–12
[48] Krunz M and Makowski A M 1998 Modeling video traffic using $M/G/\infty$ input processes: a compromise between Markovian and LRD models *IEEE J. Sel. Areas Commun.* **16** 733–48
[49] Krishnan K R and Meempat G 1997 Long-range dependence in VBR video streams and ATM traffic engineering *Perform. Eval.* **30** 45–56
[50] Bendat J S and Piersol A G 1991 *Random Data: Analysis and Measurement Procedures* 2nd edn (Singapore: Wiley)
[51] Todorovic P 1992 *An Introduction to Stochastic Processes and Their Applications* (New York: Springer)